

ITCS2175 HW 5 Answer Key

There are 69 points total over the entire homework.

Section 1.8

1) [3 points each]

Determine whether each of the following functions from the set $\{a,b,c,d\}$ to itself is one-to-one.

- a. $f(a)=b, f(b)=a, f(c)=c, f(d)=d$
 - b. $f(a)=b, f(b)=b, f(c)=d, f(d)=c$
 - c. $f(a)=d, f(b)=b, f(c)=c, f(d)=d$
-
- a. YES
 - b. NO ($f(a) = f(b) = b$)
 - c. NO ($f(a) = f(d) = d$)

2) [2.5 points each]

Determine whether each of the following functions from \mathbf{Z} to \mathbf{Z} is one-to-one.

- a. $f(n) = n - 1$
 - b. $f(n) = n^2 + 1$
 - c. $f(n) = n^3$
 - d. $f(n) = \lceil n/2 \rceil$
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- a. YES. $f(n) = n - 1$
 - b. NO. $f(n) = n^2 + 1$
 - c. YES. $f(n) = n^3$
 - d. NO. $f(n) = \lceil n/2 \rceil$

3) [5 points each]

Find $f \circ g$ and $g \circ f$ where $f(x) = x^2 + 1$ and $g(x) = x + 2$ are functions from \mathbf{R} to \mathbf{R} .

$$f \circ g = (x + 2)^2 + 1 = x^2 + 4x + 5$$

$$g \circ f = (x^2 + 1) + 2 = x^2 + 3$$

4) [10 points each]

Let f be a function from the set A to the set B . Let S and T be subsets of A . Show that:

$$\mathbf{a) } f(S \cup T) = f(S) \cup f(T)$$

Proof: We need to show that $f(S \cup T) \subseteq f(S) \cup f(T)$ and $f(S) \cup f(T) \subseteq f(S \cup T)$. First, suppose $b \in f(S \cup T)$. Then $b = f(a)$ for some element, a , in $S \cup T$. Since a is in the union of S and T , then either $a \in S$ or $a \in T$, or a is in both. Therefore $b \in f(S)$ or $b \in f(T)$ or both, so $b \in f(S) \cup f(T)$.

Now suppose that $b \in f(S) \cup f(T)$. Then either $b \in f(S)$ or $b \in f(T)$ or b is in both $f(S)$ and $f(T)$. This means that either $b = f(a)$ for some element, a , in S or some element, a , in T or in both. In all cases, a must be in the union of S and T , so $b = f(a) \in f(S \cup T)$.

Since $f(S \cup T) \subseteq f(S) \cup f(T)$ and $f(S) \cup f(T) \subseteq f(S \cup T)$, $f(S \cup T) = f(S) \cup f(T)$.

b) $f(S \cap T) \subseteq f(S) \cap f(T)$

Proof: First, suppose $b \in f(S \cap T)$. Then $b = f(a)$ for some element, a , in $(S \cap T)$. Since a is in the intersection of S and T , then a is in both S and T . Therefore b is in the intersection of $f(S)$ and $f(T)$. So $b \in f(S) \cap f(T)$, which implies that $f(S \cap T) \subseteq f(S) \cap f(T)$.

5) [10 points each]

Let f be a function from A to B . Let S and T be subsets of B . Show that:

a) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$

Proof: We have to prove that $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$ and $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$.

First, suppose $y \in f^{-1}(S \cup T)$. Then $y = f^{-1}(x)$ for some element x in $(S \cup T)$. Therefore $x = f(y)$ (by the definition of inverse functions) must be an element of S or an element of T or both. If $x \in S$, then $y \in f^{-1}(S)$ and therefore $y \in f^{-1}(S) \cup f^{-1}(T)$. By the same argument, if $x \in T$, then $y \in f^{-1}(T)$ and therefore $y \in f^{-1}(S) \cup f^{-1}(T)$. Therefore $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$.

Second, let $y \in f^{-1}(S) \cup f^{-1}(T)$. Then y is either an element of $f^{-1}(S)$ or $f^{-1}(T)$ or both. Therefore $x = f(y)$ must be in S or T or both. Thus $x \in S \cup T$, so $y \in f^{-1}(S \cup T)$. Therefore $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$.

Since $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$ and $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$, we know that $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.

b) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

Proof: We have to prove that $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$ and $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$.

First, suppose $y \in f^{-1}(S \cap T)$. Then $y = f^{-1}(x)$ for some element x in $(S \cap T)$. Therefore $x = f(y)$ must be an element of both S and T . Thus both $y \in f^{-1}(S)$ and $y \in f^{-1}(T)$, so y must be in their intersection. Therefore $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$.

Second, let $y \in f^{-1}(S) \cap f^{-1}(T)$. Then y is in both $f^{-1}(S)$ and $f^{-1}(T)$. Therefore $x = f(y)$ must be in both S and T and is thus in their intersection. So $y \in f^{-1}(S \cap T)$. Thus $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$.

Since $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$ and $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$, we have $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.