ITCS 2175 HW 3 Key

1) $(p \to q) \to r$ [10 points]

p	q	r	$p \to q$	$(p \to q) \to r$
T	**T**	**T**	**T**	**T**
T	T	F	T	F
T	**F**	**T**	**F**	**T**
T	**F**	**F**	**F**	**T**
F	**T**	**T**	**T**	**T**
F	T	F	T	F
F	**F**	**T**	**T**	**T**
F	F	F	T	F

There are 5 minterms, giving the following disjunctive normal form:
$(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$.

2) [20 points]
Use an indirect proof to prove that if 3n + 2 is even, then n is even.
The contrapositive of the statement is "If n is odd, then 3n + 2 is odd." Assume n is odd. Then $\exists k \in Z$ such that n = 2k + 1. It follows that 3n + 2 = 3(2k + 1) + 2 = 6k + 3 + 2 = 6k + 5 = (6k + 4) + 1 = 2(3k + 2) + 1. 3k + 2 is an integer, therefore 3n + 2 is odd by the definition of an odd integer. Since we have proven the contrapositive, the original statement, " if 3n + 2 is even, then n is even" is also true.

3). [20 points]
Prove that the product of two rational numbers is rational.
Let s,t be rational numbers. Then $\exists(a,b,c,d) \in Z$ where $b,d \neq 0$ such that s = a/b and t = c/d. It follows that s*t = (a/b)*(c/d) = (ac/bd). Then s*t is rational because ac, bd $\in Z$ and $b*d \neq 0$. Thus the product of two rational numbers is rational.

4) [30 points]
Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.
We will use a proof by contradiction, assuming that the product of a nonzero rational number and an irrational number is rational. Let x be a nonzero rational number and y be an irrational number. Then x = a/b where a,b $\in Z$ and $a,b \neq 0$. By our assumption, x*y is rational and so x*y = c/d where c,d $\in Z$ and $d \neq 0$. It follows that x*y = (a/b)*y = c/d. Solving for y, we get y = (cb)/(da). However, cb,da $\in Z$ and $da \neq 0$, which means that we have a rational number equal to an irrational number--a contradiction. Thus our assumption was false and x*y must be irrational, proving that the product of a nonzero rational number and an irrational number is irrational.

5) [20 points]

Prove or disprove that $\forall n \in Z \geq 0$, $2^n + 1$ is prime.

If we can find at least one counterexample, then the statement is false. A counterexample is $2^3 + 1 = 9 = 3*3$, which is composite.

6) [40 points]

Prove that the square of an integer not divisible by 5 leaves a remainder of 1 or 4 when divided by 5.

We can use a proof by cases, considering the cases that correspond to the possible remainders when an integer is divided by 5. This yields 5 cases for an integer n. n = 5k, n = 5k +1, n = 5k+2, n = 5k+3, n = 5k+4, where $k \in Z$. To complete the proof, we will look at each case individually.

Case 1: n = 5k

For this case, n is divisible by 5 and is not considered.

Case 2: n = 5k + 1

For some integer k, n = 5k + 1. It follows that
$n^2 = (5k + 1)^2 = 25k^2 + 10k + 1 = 5(5k^2 + 2k) + 1$. $5k^2 + 2k$ is an integer, so when n^2 is divided by 5, there is a remainder of 1.

Case 3: n = 5k + 2

For some integer k, n = 5k + 2. It follows that
$n^2 = (5k + 2)^2 = 25k^2 + 20k + 1 = 5(5k^2 + 4k) + 4$. $5k^2 + 4k$ is an integer, so when n^2 is divided by 5, there is a remainder of 4.

Case 4: n = 5k + 3

For some integer k, n = 5k + 3. It follows that
$n^2 = (5k + 3)^2 = 25k^2 + 30k + 9 = 5(5k^2 + 6k + 1) + 4$. $5k^2 + 6k + 1$ is an integer, so when n^2 is divided by 5, there is a remainder of 4.

Case 5: n = 5k + 4

For some integer k, n = 5k + 4. It follows that
$n^2 = (5k + 4)^2 = 25k^2 + 40k + 16 = 5(5k^2 + 8k + 3) + 1$. $5k^2 + 8k + 3$ is an integer, so when n^2 is divided by 5, there is a remainder of 1.

Thus we have shown that the square of an integer not divisible by 5 leaves a remainder of 1 or 4 when divided by 5.

7) [10 points per part--20 total]

Prove that if n is a positive integer, then n is even if an only if 7n + 4 is even.

Let p be the proposition "n is even" and q be the proposition "7n + 4 is even". We need to prove $p \leftrightarrow q$. We divide this into two steps, first proving $p \rightarrow q$ using a direct proof, then proving $q \rightarrow p$ using an indirect proof.

Part 1: Prove that if n is even then 7n + 4 is even.
Assume n is even. Then $\exists k \in Z+$ such that n = 2k. It follows that 7n + 4 = 7(2k) + 4 = 14k + 4 = 2(7k + 2). 7k + 2 is an integer, so 7n + 4 is even.

Part 2: Prove that if 7n + 4 is even, n is even.
The contrapositive of this statement would be, "if n is odd, then 7n + 4 is odd". Assume that n is odd. Then $\exists k \in Z+$ such that n = 2k + 1. It follows that 7n + 4 = 7(2k + 1) + 4 = 14k + 7+4 = 14k + 11 = (14k +10) +1 = 2(7k + 5) +1. 7k + 5 is an integer. Therefore 7n + 4 is odd, which shows that if 7n + 4 is even, n is even.