Quiz 3
ITCS 2175
(20 points each problem; choose 5)

Name ______________________

1. Find a closed form solution for the recursive definition $T(n) = 2T(n-1) + 6$ where $T(1) = 5$ and n is a positive integer. Use mathematical induction to prove that your answer is correct.

$T(1) = 5$
$T(2) = 2T(1) + 6 = 2*5 + 6$
$T(3) = 2T(2) + 6 = 2(2*5+6)+6 = 4*5+3*6$
$T(4) = 2T(3) + 6 = 2(4*5+3*6) + 6 = 8*5 + 7*6$
$T(5) = 2T(4) + 6 = 2(8*5+ 7*6) + 6 = 16*5 + 15*6$

Guess that $T(n) = 2^{n-1}*5 + (2^{n-1} – 1)*6$

PROOF:
Basis Step:
$T(1) = 2^{1-1}*5 + (2^{1-1} – 1)*6 = 1*5 + 0*6 = 5$
3 points for basis step

Inductive Step:
Assume that $T(n) = 2^{n-1}*5+ (2^{n-1} – 1)*6$
We must show that this implies that $T(n+1) = 2^{n+1-1}*5 + (2^{n+1-1} – 1)*6$
$= 2^n*5 + (2^n–1)*6$

$T(n+1) = 2T(n) + 6$ by definition
$= 2 [2^{n-1}*5 + (2^{n-1} – 1)*6] + 6$ by the inductive hypothesis
$= 2^n*5 + 2^n*6 – 2*6 + 6$
$= 2^n*5 + 2^n*6 – 6$
$=2^n*5 + (2^n– 1)*6$

2. Suppose the relation R is defined on the set Z of all integers where aRb means that $ab \leq 0$. Determine whether R is reflexive, transitive, or symmetric. Determine whether R is an equivalence relation on Z. If R is not an equivalence relation, explain why.

A relation is an equivalence relation if it is reflexive, symmetric and transitive.
R is not reflexive since for a∈Z , a*a is always positive.
R is not transitive, since a,b,c ∈Z. a,c < 0 and b>0 , then aRb, bRc but it is not true that aRc.
R is symmetric, since for a,b ∈Z if a,b < 0, then b,a < 0.

3. Suppose you have a class with 25 students – 10 freshmen, 8 sophomores, and 7 juniors. In how many ways can you:

   - Put all 25 in a line?

   - Put all students in a line so that the freshmen are first, the sophomores are in the middle, and the juniors are at the end?

   - Get a committee of 7 people?

P(25,25) = 25!/(25-25)! = 25!
P(10,10)P(8,8)P(7,7) = 10!8!7!
C(25,7) = 25!/7!18! = 3*25*24*23*22*20*19 (This simplified form isn't necessary)

4. A professor teaching a discrete math course give a multiple choice quiz that has 9 questions, each with 4 possible responses: a,b,c,d. What is the minimum number of students that must be in the class in order to guarantee that at least three answer sheets must be identical? (Assume that no answers are left blank).

This can be set up as a Pigeonhole Problem. Each student must do 9 questions for which there are 4 possible responses per questions. The total number of boxes is thus 4^9. To insure that at least one box has at least 3 students in it, there must be at minimum $2*4^9 + 1$ students.

5. Let $a_1 = 2$, $a_2 = 9$, and $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$. Prove that $a_n \leq 3^n$ for all positive integers n.

Proof by (Strong) Induction

Basis Step:
$a_3 = 2a_2 + 3a_1 = 24 \leq 3^3$.
The theorem is true for n = 3.

Inductive Step.
Assume that $a_k \leq 3^k$ for $3 \leq k < n$.
Then we must show that this implies $a_n \leq 3^n$.

$$a_n = 2a_{n-1} + 3a_{n-2} \leq 2*3^{n-1} + 3*3^{n-2}$$
$$= 2*3^{n-1} + 3^{n-1} = 3*3^{n-1} = 3^n.$$

We can see by inspection that the theorem holds for n = 1 and n =2, thus we have shown that $a_n \leq 3^n$ for all positive integers n.

6. Given the relation on the set {0,2,4,6} defined by the ordered pairs: (0,0), (0,4), (2,2), (4,0), (4,4), (6,2), (6,4), (6,6).
   - What is the zero-1 matrix?

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

   - Is R reflexive? Justify your answer.

Yes, (0,0), (2,2), (4,4), (6,6) all appear in the relation.

   - Is R symmetric? Justify your answer.

No, (6,2) is in the relation (and also (6,4)), but the transpose is not.

   - Is R transitive? Justify your answer.

No, if (a,b) and (b,c) are in the relation, (a,c) is, too. However, (6,4) and (4,0) are in the relation, but not (6,0).

   - Is R an equivalence relation? Explain why.

A relation is an equivalence relation if it is reflexive, symmetric and transitive. Since the above is not symmetric and not transitive, it is not an equivalence relation